MATHEMATICS

# HOMOTOPY THEORY OF PRODUCTS ON SPHERES. I.

BY

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(Communicated by Prof. H. FREUDENTHAL at the meeting of January 25, 1969)

## INTRODUCTION

In this thesis we study some special products on spheres from the standpoint of homotopy theory.

§§ 6–9 contain our main results. A survey of some results on reflecting products is found in theorem (9.21).

The map $Gf$, as defined in (4.2), is commonly known under the name "Hopf construction", although this name is rather meaningless. Therefore we propose to call it "Hopf suspension".

When considering $S^n$, we shall always exclude the case $n=0$.

By $Z$ we denote the group of integers, and by $Z_m$ we denote the group of integers modulo $m$.

The symbol $\cong$ denotes that two groups are isomorphic.

The end of a proof is denoted by Q.E.D.

Numbers between square brackets refer to the references.

SP 1.2.3 stands for: E. H. Spanier, "Algebraic Topology", chapter 1, section 2, item 3.

## 1. PRELIMINARIES

By $(x_1, x_2, \ldots, x_r)$ we denote the point $(y_1, y_2, \ldots, y_k, \ldots)$ in Hilbert space $R$, for which $y_k = x_k$ if $1 \leqslant k \leqslant r$ and $y_k = 0$ if $k > r$.

Let $S^r$ be the subspace of $R$, which consists of the points $(x_1, x_2, \ldots, x_{r+1})$ such that $x_1^2 + x_2^2 + \ldots + x_{r+1}^2 = 1$.

$E^r_+$, $E^r_- \subset S^r$ are the hemispheres in which $x_{r+1} \geqslant 0$, $x_{r+1} \leqslant 0$.

The $r$-dimensional cube $I^r \subset R$ consists of the points $(x_1, x_2, \ldots, x_r)$ such that $-1 \leqslant x_i \leqslant 1$, $i = 1, 2, \ldots, r$.

By $\dot{I}^r$ we denote the boundary of $I^r$.

In all these spaces the basepoint will be the point

$$e = (-1, 0, \ldots, 0, \ldots).$$

We shall orient $I^r$ $(r \geqslant 1)$ as follows.

Let the points $a^{\langle i \rangle}$, $i = 0, 1, 2, \ldots, r$, be given by

$$a^{\langle i \rangle} = (a_1^{\langle i \rangle}, a_2^{\langle i \rangle}, \ldots, a_r^{\langle i \rangle}).$$

Then the simplex $[a^{\langle 0\rangle}, a^{\langle 1\rangle}, \ldots, a^{\langle r\rangle}]$ is positively oriented if and only if $(\det\{a_j^{\langle i\rangle} - a_j^{\langle 0\rangle};\ 1 \leqslant i \leqslant r,\ 1 \leqslant j \leqslant r\}) > 0$.

We choose the orientation of $\dot{I}^r$ so, that $\dot{I}^r$ is the combinatorial boundary of $I^r$.

Let the map $\varrho\colon S^{r-1} \to \dot{I}^r$ be the radial projection.

The orientation of $S^{r-1}$ is such that the map $\varrho$ has degree $+1$.

Let $I$ denote the interval $-1 \leqslant t \leqslant 1$.

We shall often need a map

$$d_r\colon S^r \times I \to S^{r+1}$$

which has the following properties:

(i) $d_r$ maps $S^r \times (I - \dot{I})$ homeomorphically onto $S^{r+1} - e$,
(ii) $d_r(S^r \times \dot{I} \cup e \times I) = e$,
(iii) $d_r(x, 0) = x$ for all $x \in S^r$,
(iv) $d_r(x, t) \in E_+^{r+1}$ if $t \geqslant 0$ and $d_r(x, t) \in E_-^{r+1}$ if $t \leqslant 0$, for all $x \in S^r$, $t \in I$.

Another important map is

$$\Phi_r\colon (I^r, \dot{I}^r) \to (S^r, e)$$

which is defined inductively by

$$\Phi_1(x) = (\cos \pi x, \sin \pi x), \quad x \in I^1,$$

$$\Phi_{n+1}(x_1, \ldots, x_n, x_{n+1}) = d_n(\Phi_n(x_1, \ldots, x_n), x_{n+1}).$$

Notice that $\Phi_r$ maps $I^r - \dot{I}^r$ homeomorphically onto $S^r - e$, and that $\Phi_r$ has degree $+1$.

It is very useful to represent elements of $\pi_r(X)$, where $X$ is a space, in different ways. Usually one represents them by maps

$$(1.1) \qquad \begin{aligned} f&\colon (I^r, \dot{I}^r) \to (X, x_0) \\ g&\colon (S^r, e) \to (X, x_0) \\ h&\colon (\dot{I}^{r+1}, e) \to (X, x_0) \end{aligned}$$

where $x_0$ is the basepoint of the space $X$.

We shall connect these representations by the following convention:

(1.2) Let $\varrho$ be the radial projection from $S^r$ onto $\dot{I}^{r+1}$, and let $f$, $g$, $h$ be maps as in (1.1).
If $f = g\Phi_r$ and $g = h\varrho$, then $f$, $g$, $h$ represent the same element of $\pi_r(X)$.

When we consider a map or a homotopy, from a space $X$ into a space $Y$, then we always assume that basepoints are respected.

Let $X_1$, $X_2$ and $Y_1$, $Y_2$ be subspaces of $X$ and $Y$.

Then by the notations

$$(1.3)\qquad \begin{aligned} f&: (X, X_1, X_2) \to (Y, Y_1, Y_2) \\ g&: (X, X_1) \qquad \to (Y, Y_1) \end{aligned}$$

we do not only mean that $f(X_1) \subset Y_1$, $f(X_2) \subset Y_2$, $g(X_1) \subset Y_1$, but also that these inclusions stay valid during homotopies of $f$ and $g$.

## 2. The Freudenthal suspension

Consider a map $f: S^n \to S^r$.
Then there exists a map

$$Ef: S^{n+1} \to S^{r+1}$$

which has the following properties:

(2.1) (i) $Ef$ and $f$ agree on $S^n$,
(ii) $Ef\,(E^{n+1}_+) \subset E^{r+1}_+$ and $Ef\,(E^{n+1}_-) \subset E^{r+1}_-$.

It is well-known that, given $f$, all maps $S^{n+1} \to S^{r+1}$ which fulfil conditions (i) and (ii) of (2.1) are homotopic.

The *suspension homomorphism*

$$E: \pi_n(S^r) \to \pi_{n+1}(S^{r+1})$$

is now defined as follows: Let $f: S^n \to S^r$ represent $\alpha \in \pi_n(S^r)$, then $Ef: S^{n+1} \to S^{r+1}$ will be a representative map for the element $E\alpha \in \pi_{n+1}(S^{r+1})$.

By $[\alpha, \beta]$ we denote the Whitehead product between elements $\alpha \in \pi_p(X)$, $\beta \in \pi_q(X)$. For the definition of this product see § 3 of [19], where it is denoted by $\alpha \cdot \beta$.

Let $\iota_n$ be the positive generator of $\pi_n(S^n)$.

In [5] and [6] there occurs the sequence

$$(2.2)\qquad \pi_i(S^n) \xrightarrow{E} \pi_{i+1}(S^{n+1}) \xrightarrow{H'} \pi_{i-n}(S^n) \xrightarrow{P} \pi_{i-1}(S^n) \xrightarrow{E} \cdots$$

which is exact if $i \leqslant 3n-2$. In this sequence $H'$ is a generalized Hopf invariant homomorphism, and $P$ is the homomorphism defined by: $P\alpha = [\alpha, \iota_n]$, $\alpha \in \pi_{i-n}(S^n)$.

By Corollary (4.5) and Theorem (4.24) of [6], Corollary 2 of [5], and by the second corollary on page 247 of [17], we have for $k \geqslant 1$:

(2.3) $E: \pi_{2n}(S^n) \to \pi_{2n+1}(S^{n+1})$ is injective if $n=2$ or $n=4k-1$,
$E: \pi_{2n-1}(S^{n-1}) \to \pi_{2n}(S^n)$ is onto if $n=4k+2$ or $n=4k+1$.

By the exactness of (2.2) and by the fact that $\pi_{n+1}(S^n) \equiv Z_2$ for $n \geqslant 3$ (see [3]), we have in addition to (2.3):

(2.4) If $n=4k$, then $E: \pi_{2n}(S^n) \to \pi_{2n+1}(S^{n+1})$ is not injective and $E: \pi_{2n-1}(S^{n-1}) \to \pi_{2n}(S^n)$ is not onto.

(2.5) **Lemma**

Let $\iota_n$ be the positive generator of $\pi_n(S^n)$, and let $\alpha$ be any element of $\pi_{2n}(S^n)$.

If $n$ is *not* a multiple of 4, we have in $\pi_{2n}(S^n)$:

$$(k\iota_n) \circ \alpha = k\alpha \text{ for every integer } k.$$

PROOF: If $n$ is not a multiple of 4, then it follows from (2.3) that either $\alpha$ is a suspension element, or $E: \pi_{2n}(S^n) \to \pi_{2n+1}(S^{n+1})$ is injective. (The case $n=1$ is fully trivial). If $\alpha$ is a suspension, then

$$(k\iota_n) \circ \alpha = k(\iota_n \circ \alpha) = k\alpha.$$

If $E$ is injective, we form

$$E((k\iota_n) \circ \alpha - k\alpha) = k\iota_{n+1} \circ E\alpha - kE\alpha = kE\alpha - kE\alpha = 0.$$

Q.E.D.

3. SEPARATION ELEMENTS

The relative homotopy group $\pi_r(X, A)$, where $A$ is a subspace of $X$, is defined to be the set of homotopy classes of maps

$$(I^r, \dot{I}^r, J^{r-1}) \to (X, A, x_0),$$

where $x_0 \in A$ is the basepoint of $X$, and $J^{r-1}$ is the subspace of $\dot{I}^r$ in which $x_r > -1$.

Every map $g: (I^r, \dot{I}^r, e) \to (X, A, x_0)$ is homotopic to a map

$$g': (I^r, \dot{I}^r, J^{r-1}) \to (X, A, x_0).$$

Now let $\Sigma$ be the subspace of $S^p \times S^q$, defined by

$$\Sigma = S^p \times e \cup e \times S^q.$$

We orient $S^p \times S^q$ so, that the map

$$\Phi_{p,q}: (I^{p+q}, \dot{I}^{p+q}) \to (S^p \times S^q, \Sigma)$$

has degree $+1$. Here $\Phi_{p,q}$ is defined by

(3.1) $\Phi_{p,q}(x_1, \ldots, x_p, x_{p+1}, \ldots, x_{p+q}) = (\Phi_p(x_1, \ldots, x_p), \Phi_q(x_{p+1}, \ldots, x_{p+q}))$.

The positive generator $\gamma$ of $\pi_{p+q}(S^p \times S^q, \Sigma)$ is then represented by a homotopic map

(3.2) $$\Phi'_{p,q}: (I^{p+q}, \dot{I}^{p+q}, J^{p+q-1}) \to (S^p \times S^q, \Sigma, e \times e).$$

Now consider two maps $f, g: (S^p \times S^q, e \times e) \to (X, x_0)$ and assume that *$f$ and $g$ are the same on $\Sigma$.*

Then we define a map

$$D(f, g): (I^{p+q}, \dot{I}^{p+q}) \to (X, x_0)$$

by the formula

$$\text{(3.3)} \quad \begin{aligned} D(f, g)(x_1, \ldots, x_{r-1}, x_r) &= f\Phi'_{p,q}(x_1, \ldots, x_{r-1}, 2x_r - 1) \text{ if } x_r \geqslant 0, \\ &= g\Phi'_{p,q}(x_1, \ldots, x_{r-1}, -2x_r - 1) \text{ if } x_r \leqslant 0, \end{aligned}$$

where $r = p+q$. The *separation element* $d(f, g) \in \pi_{p+q}(X)$ is now defined to be that element of $\pi_{p+q}(X)$ which is represented by $D(f, g)$.

For simplicity, let us assume that $X$ is $n$-simple for all $n$ (for example $X$ arcwise connected and $\pi_1(X) = 0$), so the choice of the basepoint of $X$ is immaterial.

Then, starting from (3.3), one easily sees that we can represent $d(f, g)$ also by a map

$$D'(f, g): S^{p+q} \to X$$

as follows. Take a map $u: (E_+^{p+q}, S^{p+q-1}) \to (I^{p+q}, \dot{I}^{p+q})$ of degree $+1$, and a map $v: (E_-^{p+q}, S^{p+q-1}) \to (I^{p+q}, \dot{I}^{p+q})$ of degree $-1$, such that $u$ and $v$ agree on $S^{p+q-1}$.

Next we define $D'(f, g)$ by

$$\text{(3.4)} \quad \begin{aligned} D'(f, g)(x) &= f\Phi_{p,q}u(x) \text{ if } x \in E_+^{p+q}, \\ &= g\Phi_{p,q}v(x) \text{ if } x \in E_-^{p+q}. \end{aligned}$$

If we identify $S^{p+q}$ and $\dot{I}^{p+q+1}$ by the radial projection $\varrho$ as in (1.2), then (3.4) would be a very useful formula if instead of $u$ and $v$ we could take the orthogonal projections along the $x_{p+q+1}$-axis from $\varrho E_+^{p+q}$ and from $\varrho E_-^{p+q}$ onto $I^{p+q}$.

However, these projections have degree $(-1)^{p+q}$ and $(-1)^{p+q+1}$ respectively.

For future purpose we define the map

$$D''(f, g): \dot{I}^{p+q+1} \to X$$

by the formula

$$\text{(3.5)} \quad \begin{aligned} D''(f, g)(x_1, \ldots, x_r, x_{r+1}) &= f\Phi_{p,q}(x_1, \ldots, x_r) \text{ if } x_{r+1} \geqslant 0, \\ &= g\Phi_{p,q}(x_1, \ldots, x_r) \text{ if } x_{r+1} \leqslant 0, \end{aligned}$$

where $r = p+q$.
Because of our comment on (3.4), we have:

(3.6) **Lemma**

The map $D''(f, g): \dot{I}^{p+q+1} \to X$, as defined in (3.5), represents the element $(-1)^{p+q} d(f, g)$ of $\pi_{p+q}(X)$.

Now we supply a survey of the main properties of the separation element as these are listed by I. M. James in the appendix of [9], but we will write them down in the form we need later on. In this survey $L$ stands for $S^p \times S^q$ and $X$, $Y$ are spaces.

(3.7) Let $f, g: L \to X$ be maps which agree on $\Sigma$. Then $f$ is homotopic to $g$, relative to $\Sigma$, if and only if $d(f, g) = 0$.

(3.8) Let $f, g, h: L \to X$ be maps which agree on $\Sigma$. Then $d(f, g) = = d(f, h) + d(h, g)$.

(3.9) Let an element $\delta \in \pi_{p+q}(X)$ and a map $f: L \to X$ be given. Then there exists a map $g: L \to X$, which agrees with $f$ on $\Sigma$, such that $d(f, g) = \delta$.

(3.10) Let $f_t, g_t: L \to X$ be homotopies which agree on $\Sigma$, where $0 \leqslant t \leqslant 1$. Then $d(f_0, g_0) = d(f_1, g_1)$.

(3.11) Let $f, g: L \to X$ be maps which agree on $\Sigma$. For a map $h: X \to Y$, let $h_*: \pi_{p+q}(X) \to \pi_{p+q}(Y)$ be the homomorphism induced by $h$. Then $d(hf, hg) = h_* d(f, g)$.

(3.12) Let $k: (L, \Sigma) \to (L, \Sigma)$ be a map of degree $r$. Let $f, g: L \to X$ be maps which agree on $\Sigma$. Then $d(fk, gk) = rd(f, g)$.

Property (3.12) calls for the following remark.

For the map $k: (L, \Sigma) \to (L, \Sigma)$, let $k'_*: H_{p+q}(L, \Sigma) \to H_{p+q}(L, \Sigma)$ be the induced homomorphism for integral relative homology groups. As the map $\Phi_{p,q}: (I^{p+q}, \dot{I}^{p+q}) \to (L, \Sigma)$ is a homeomorphism from $I^{p+q} - \dot{I}^{p+q}$ onto $L - \Sigma$, the integral relative homology groups $H_{p+q}(I^{p+q}, \dot{I}^{p+q})$ and $H_{p+q}(L, \Sigma)$ are isomorphic (see SP 4.8.9). Therefore we have:

$$H_m(L, \Sigma) \equiv H_m(I^m, \dot{I}^m) \equiv Z, \text{ where } m = p + q.$$

Let $\alpha$ be a generator of $H_{p+q}(L, \Sigma)$, then the degree of the map $k$ is said to be $r$, if $k'_* \alpha = r\alpha$.

In the exact homology sequence of the pair $(L, \Sigma)$ applies:

$$j_*: H_{p+q}(L) \to H_{p+q}(L, \Sigma) \text{ is injective,}$$

since $H_{p+q}(\Sigma) = 0$.

It is well-known that $H_{p+q}(L) \equiv Z$, with generator $\iota_p \otimes \iota_q$, the latter being the tensor product between generators $\iota_p$ and $\iota_q$ of $H_p(S^p)$ and $H_q(S^q)$ respectively.

Let $k''_*: H_{p+q}(L) \to H_{p+q}(L)$ be the homomorphism induced by the map $k$, and suppose that $k$ has degree $r$.

Then we have: $k''_*(\iota_p \otimes \iota_q) = r(\iota_p \otimes \iota_q)$.

(3.13) **Theorem**

Let $u: S^p \to S^p$ and $v: S^q \to S^q$ be maps of degree $m$ and $n$ respectively. Then the map $k$: $(L, \Sigma) \to (L, \Sigma)$ defined by

$$k(x, y) = (u(x), v(y)) \quad x \in S^p,\ y \in S^q$$

has degree $m \cdot n$.

PROOF: $k''_*(\iota_p \otimes \iota_q) = m\iota_p \otimes n\iota_q = m \cdot n(\iota_p \otimes \iota_q)$. Q.E.D.

We want to specialize property (3.11) in case $X$, $Y$ are spheres:

(3.14) Let $f, g$: $L \to S^r$ be maps which agree on $\Sigma$, and suppose that the map $h: S^r \to S^n$ represents $\alpha \in \pi_r(S^n)$. Then $d(hf, hg) = \alpha \circ d(f, g)$.

In the appendix of [9], I. M. James defines the separation element up to a sign. Using our definition of the separation element, it is easily verified that the results of [9] are valid exactly, without changes of signs.

## 4. THE HOPF SUSPENSION

In the usual way we identify $I^{m+n}$ with $I^m \times I^n$ and we consider $I^m \times \dot{I}^n$, $\dot{I}^m \times I^n$, $\dot{I}^m \times \dot{I}^n$ the corresponding subspaces of $\dot{I}^{m+n}$.

We describe every point of $\dot{I}^{p+q+2}$ by three co-ordinates $(x, y, t)$, $x \in S^p$, $y \in S^q$, $t \in I$, such that

(4.1) $(x, y, t)$ denotes the point

$$((1-t)\cdot\varrho(x), \varrho(y)) \in I^{p+1} \times \dot{I}^{q+1} \text{ when } t \geqslant 0,$$

$$(\varrho(x), (1+t)\cdot\varrho(y)) \in \dot{I}^{p+1} \times I^{q+1} \text{ when } t \leqslant 0,$$

where $\varrho$ is again the radial projection as in (1.2).

Notice that the correspondence (4.1) between $S^p \times S^q \times I$ and $\dot{I}^{p+q+2}$ is homeomorphically on $S^p \times S^q \times (I - \dot{I})$.

For any map

$$f: S^p \times S^q \to S^r$$

we construct its *Hopf suspension*

$$Gf: \dot{I}^{p+q+2} \to S^{r+1}$$

by the formula

(4.2) $$Gf(x, y, t) = d_r(f(x, y), t) \quad x \in S^p,\ y \in S^q,\ t \in I.$$

Let the map $\bar{f}$: $\dot{I}^{p+1} \times \dot{I}^{q+1} \to S^r$ be defined by

$$\bar{f}(\varrho(x), \varrho(y)) = f(x, y) \qquad x \in S^p,\ y \in S^q.$$

Then we observe the following properties of the map $Gf$:

(i) $Gf$ agrees with $\bar{f}$ on the subspace $\dot{I}^{p+1} \times \dot{I}^{q+1}$ of $\dot{I}^{p+q+2}$,

(ii) $Gf(I^{p+1} \times \dot{I}^{q+1}) \subset E_+^{r+1}$ and $Gf(\dot{I}^{p+1} \times I^{q+1}) \subset E_-^{r+1}$.

Let $f: S^p \times S^q \to S^r$ be given, then any map $\dot{I}^{p+q+2} \to S^{r+1}$ which fulfils properties (i) and (ii) above is homotopic to $Gf$.

Let $\{Gf\}$ denote the homotopy class of $Gf$ in $\pi_{p+q+1}(S^{r+1})$.

Instead of $\{Gf\}$ we will often choose the element $c(f) \in \pi_{p+q+1}(S^{r+1})$, defined by

$$c(f) = (-1)^{q+1}\{Gf\}, \tag{4.3}$$

according to Theorem (8.5) of [8].

Because of the last remark of § 3, we may copy Corollary (9.2) of [9] in the form we need:

(4.4) **Lemma**

Let $f, g: S^p \times S^q \to S^r$ be maps which agree on $\Sigma$. Then we have in $\pi_{p+q+1}(S^{r+1})$:

$$c(g) - c(f) = Ed(f, g).$$

A second important property of the Hopf suspension is

(4.5) If $f, g: S^p \times S^q \to S^r$ are homotopic maps, then $c(f) = c(g)$.

The type of a map $f: S^p \times S^q \to S^r$ is the pair $(\alpha, \beta)$, where $\alpha \in \pi_p(S^r)$ and $\beta \in \pi_q(S^r)$ are the homotopy classes of the sections $f': S^p \to S^r$ and $f'': S^q \to S^r$, which are defined by $f'(x) = f(x, e)$ and $f''(y) = f(e, y)$ respectively.

(4.6) (see Theorem (2.24) of [8])

Suppose we have maps $g: S^i \to S^p$ and $h: S^j \to S^q$.

Let $\alpha \in \pi_i(S^p)$ and $\beta \in \pi_j(S^q)$ denote the homotopy classes of $g$ and $h$.

Let $f': S^i \times S^j \to S^r$ be the composition of a given map $f: S^p \times S^q \to S^r$ and the product map $g \times h: S^i \times S^j \to S^p \times S^q$.

Then $c(f') = c(f) \circ (\alpha \star \beta)$, where

$$\alpha \star \beta = (-1)^{p(j+q)} E^{p+1}\beta \circ E^{j+1}\alpha = (-1)^{i(j+q)} E^{q+1}\alpha \circ E^{i+1}\beta.$$

An easy consequence of (4.6) is

(4.7) If, for a map $f: S^p \times S^q \to S^r$, $f(x, y)$ depends only on $x$, or only on $y$, then $c(f) = 0$.

(4.8) Let $f: S^p \times S^q \to S^r$ be a map, and suppose that the map $g: S^r \to S^n$ represents the element $\alpha \in \pi_r(S^n)$. Then we have in $\pi_{p+q+1}(S^{n+1})$:

$$c(gf) = E\alpha \circ c(f).$$

Remark: (4.8) follows easily from (8.3) of [8], or from (4.2) and (4.3) above.

## 5. The right suspension

In this section we follow the construction of H. Toda in § 3 of [13]. For a map $f: S^p \times S^q \to S^r$ there exists its *right suspension*

$$E'f: S^p \times S^{q+1} \to S^{r+1}$$

which has the following properties:

(5.1) (i) $E'f$ agrees with $f$ on the subspace $S^p \times S^q$ of $S^p \times S^{q+1}$,
(ii) $E'f(S^p \times E_+^{q+1}) \subset E_+^{r+1}$, $E'f(S^p \times E_-^{q+1}) \subset E_-^{r+1}$.

If the map $f: S^p \times S^q \to S^r$ is given, then all maps $S^p \times S^{q+1} \to S^{r+1}$ which fulfil conditions (i) and (ii) of (5.1) are homotopic.

In Lemma (3.2) of [13], Toda shows the connection between the right suspension and the Hopf suspension, by proving:

(5.2) For any map $f: S^p \times S^q \to S^r$, we have in $\pi_{p+q+2}(S^{r+2})$:

$$\{G(E'f)\} = -E\{Gf\}.$$

According to (4.3) and (5.2) we can write down:

(5.3) **Lemma**

For any map $f: S^p \times S^q \to S^r$ we have in $\pi_{p+q+2}(S^{r+2})$:

$$c(E'f) = Ec(f).$$

In the following theorem we show the connection between the separation element and the right suspension.

(5.4) **Theorem**

Let $f, g: S^p \times S^q \to S^r$ be maps which agree on $\Sigma$, and let $E'f, E'g$: $S^p \times S^{q+1} \to S^{r+1}$ also have the same sections.

Then we have in $\pi_{p+q+1}(S^{r+1})$:

$$d(E'f, E'g) = Ed(f, g).$$

Proof: According to (3.5) and lemma (3.6), $(-1)^{p+q+1}d(E'f, E'g)$ is represented by the map $F: \dot{I}^{p+q+2} \to S^{r+1}$, defined by

$$\begin{aligned} F(x_1, \ldots, x_{p+q}, x_{p+q+1}, x_{p+q+2}) = \\ &= E'f(\Phi_p(x_1, \ldots, x_p), \Phi_{q+1}(x_{p+1}, \ldots, x_{p+q+1})) \text{ if } x_{p+q+2} \geqslant 0, \\ &= E'g(\Phi_p(x_1, \ldots, x_p), \Phi_{q+1}(x_{p+1}, \ldots, x_{p+q+1})) \text{ if } x_{p+q+2} \leqslant 0. \end{aligned}$$

Now let $v: \dot{I}^{p+q+2} \to \dot{I}^{p+q+2}$ be the map which interchanges the last two co-ordinates, so

$$v(x_1, \ldots, x_{p+q}, x_{p+q+1}, x_{p+q+2}) = (x_1, \ldots, x_{p+q}, x_{p+q+2}, x_{p+q+1}).$$

The map $v$ has degree $-1$, thus $Fv\colon \dot{I}^{p+q+2} \to S^{r+1}$ represents

$$(-1)^{p+q}d(E'f, E'g).$$

For clearness sake we write down $Fv$ explicitly:

$$Fv(x_1, \ldots, x_{p+q}, x_{p+q+1}, x_{p+q+2}) =$$
$$= E'f(\Phi_p(x_1, \ldots, x_p), \Phi_{q+1}(x_{p+1}, \ldots, x_{p+q}, x_{p+q+2})) \text{ if } x_{p+q+1} \geqslant 0,$$
$$= E'g(\Phi_p(x_1, \ldots, x_p), \Phi_{q+1}(x_{p+1}, \ldots, x_{p+q}, x_{p+q+2})) \text{ if } x_{p+q+1} \leqslant 0.$$

Now by the definitions of $\Phi_{q+1}$, $E'f$, $E'g$ and $D''(f, g)$, we can sum up the following properties of $Fv$:

(i) $Fv(x_1, \ldots, x_{p+q+1}, 0) = D''(f, g)(x_1, \ldots, x_{p+q+1})$,
(ii) $Fv(x_1, \ldots, x_{p+q+1}, x_{p+q+2}) \in E_+^{r+1}$ if $x_{p+q+2} \geqslant 0$,
(iii) $Fv(x_1, \ldots, x_{p+q+1}, x_{p+q+2}) \in E_-^{r+1}$ if $x_{p+q+2} \leqslant 0$.

So $Fv$ satisfies the properties of $ED''(f, g)$ (see (2.1)), and from this we may conclude:

$$(-1)^{p+q}d(E'f, E'g) = (-1)^{p+q}Ed(f, g). \qquad \text{Q.E.D.}$$

## 6. Some general remarks on products on spheres

A product $\mu$ on $S^n$ will be a map $\mu\colon S^n \times S^n \to S^n$, such that $\mu(e, e) = e$. For shortness sake we shall often note down: $\mu(x, y) = x \cdot y$.

Also in this section we shall use the notations:

$$\Sigma = S^n \vee S^n = S^n \times e \cup e \times S^n.$$

The product $\mu$ on $S^n$ is said to be of type $(p, q)$, if the maps $\mu', \mu''$: $S^n \to S^n$, defined by $\mu'(x) = \mu(x, e)$ and $\mu''(x) = \mu(e, x)$, are of degree $p$, $q$ respectively.

Let $\mu$ be a product on $S^n$, and let $f, g\colon X \to S^n$ be maps. Then we define the map $f \cdot g\colon X \to S^n$ by $f \cdot g(z) = f(z) \cdot g(z)$, $z \in X$.

(6.1) **Theorem**

Let $\mu$ be a product on $S^n$, and let $f, g\colon S^m \to S^n$ be maps. For a map $h$, $\{h\}$ denotes the homotopy class of $h$.

Then in $\pi_m(S^n)$ applies:

$$\{f \cdot g\} = \{f \cdot e\} + \{e \cdot g\},$$

where $e\colon S^m \to S^n$ denotes the constant map: $e(z) = e$.

Proof: Represent $\{f\}, \{g\} \in \pi_m(S^n)$ by maps $f', g'\colon (I^m, \dot{I}^m) \to (S^n, e)$. Next consider the maps $F, G\colon (I^m, \dot{I}^m) \to (S^n, e)$ defined by

$$\begin{aligned} F(x_1, x_2, \ldots, x_m) &= e && \text{if } x_1 \geqslant 0,\\ &= f'(2x_1 + 1, x_2, \ldots, x_m) && \text{if } x_1 \leqslant 0,\\ G(x_1, x_2, \ldots, x_m) &= g'(2x_1 - 1, x_2, \ldots, x_m) && \text{if } x_1 \geqslant 0,\\ &= e && \text{if } x_1 \leqslant 0. \end{aligned}$$

Now it is clear that $F \sim f'$ and $G \sim g'$, so $F \cdot G \sim f' \cdot g'$. Let us write down $F \cdot G$: $(I^m, \dot{I}^m) \to (S^n, e)$ explicitly:

$$F \cdot G(x_1, x_2, \ldots, x_m) = e \cdot g'(2x_1 - 1, x_2, \ldots, x_m) \text{ if } x_1 \geqslant 0,$$
$$= f' \cdot e(2x_1 + 1, x_2, \ldots, x_m) \text{ if } x_1 \leqslant 0.$$

Thus $\{F \cdot G\} = \{f' \cdot e\} + \{e \cdot g'\}$, and by the final remark, which is easy to prove, that $\{f \cdot g\}$ is independent of the representation of $\{f\}$ and $\{g\}$, the theorem is proved. Q.E.D.

For maps $f, g$: $S^n \times S^n \to S^n$, let $f|\Sigma = g|\Sigma$ denote that $f$ and $g$ agree on $\Sigma$.

(6.2) **Theorem**

Let $\mu$ be a product on $S^n$ of type $(p, q)$, and let $u, v, u', v'$: $(S^n \times S^n, e \times e) \to (S^n, e)$ be maps such that $u|\Sigma = v|\Sigma$ and $u'|\Sigma = v'|\Sigma$. Finally, let $\iota_n$ be the positive generator of $\pi_n(S^n)$.

Then in $\pi_{2n}(S^n)$ we have the equation:

$$d(u \cdot u', v \cdot v') = p\iota_n \circ d(u, v) + q\iota_n \circ d(u', v').$$

Proof: Represent the separation element $d(u \cdot u', v \cdot v')$ by the map $D(u \cdot u', v \cdot v')$: $(I^{2n}, \dot{I}^{2n}) \to (S^n, e)$ as in (3.3). By simply writing down the explicit formula for $D(u \cdot u', v \cdot v')$ as in (3.3), it is immediately clear that

$$D(u \cdot u', v \cdot v') = D(u, v) \cdot D(u', v').$$

Now by theorem (6.1) we may note down:

$$\{D(u \cdot u', v \cdot v')\} = \{D(u, v) \cdot e\} + \{e \cdot D(u', v')\},$$

but $D(u, v) \cdot e = \mu' D(u, v)$ and $e \cdot D(u', v') = \mu'' D(u', v')$ (see the introduction of this section). By the fact that $\mu'$ and $\mu''$ are maps of degree $p$ and $q$, we have in $\pi_{2n}(S^n)$:

$$\{D(u \cdot u', v \cdot v')\} = p\iota_n \circ \{D(u, v)\} + q\iota_n \circ \{D(u', v')\}. \qquad \text{Q.E.D.}$$

Because of lemma (2.5) we are able to simplify the assertion of this theorem, if $n$ is not a multiple of 4:

(6.3) **Corollary**

If $n$ is *not* a multiple of 4, then the assertion in theorem (6.2) reduces to

$$d(u \cdot u', v \cdot v') = pd(u, v) + qd(u', v').$$

**Remark**: Because of (3.14) the equation in theorem (6.2) is equivalent to the equation

$$d(u \cdot u', v \cdot v') = d(u \cdot e, v \cdot e) + d(e \cdot u', e \cdot v').$$

## 7. Homotopy classification of products on spheres

By the homotopy extension theorem, every map $f\colon S_1^n \times S_2^n \to S^n$ of type $(p, q)$ is homotopic to an extension of a fixed map $g_0\colon S_1^n \vee S_2^n \to S^n$, such that $g_0|S_1^n$, $g_0|S_2^n$ have degree $p$, $q$ respectively.

Because of this fact we can apply the theory of W. D. Barcus and M. G. Barratt in [2], in particular § 5 of [2].

We provide a brief account of the methods used in [2].

Let $L$ be a $CW$ complex, and let $K$ be a subcomplex of $L$ such that $L = K \cup e^{r+1}$. Here $e^{r+1}$ is a $(r+1)$-cel, and the attaching map will be $g\colon S^r \to K$.

For simplicity we assume here that $L = S^n \times S^n$ and $K = \Sigma$, so $r = 2n-1$.

For a space $X$ let $X^K$ denote the function space of maps $(K, k_0) \to (X, x_0)$, where $k_0$, $x_0$ are basepoints.

Elements of $\pi_1(X^K, u_0)$, $u_0 \in X^K$, may be represented as homotopy classes of maps $F\colon (K \times S^1, k_0 \times S^1) \to (X, x_0)$ such that $F(k, e) = u_0(k)$.

The attaching map $g\colon S^r \to K$ induces a homomorphism

$$g^*\colon \pi_1(X^K, u_0) \to \pi_1(X^{S^r}, u_0 g).$$

For any map $v\colon S^r \to X$ there is defined a map

$$v^\flat\colon (S^r \times S^1, e \times S^1) \to (X, x_0)$$

by $v^\flat(x, y) = v(x)$.

Let the element $\{F\} \in \pi_1(X^{S^r}, v_0)$ be represented by the map

$$F\colon (S^r \times S^1, e \times S^1) \to (X, x_0),$$

then the isomorphism

$$v_0^\natural\colon \pi_1(X^{S^r}, v_0) \to \pi_{r+1}(X, x_0)$$

is defined by $v_0^\natural\{F\} = d(F, v_0^\flat)$.

Let $\alpha \in \pi_r(K)$ be the homotopy class of the attaching map $g$, then there is defined a homomorphism

$$\alpha_{u_0}\colon \pi_1(X^K, u_0) \to \pi_{r+1}(X, x_0)$$

by the formula: $\alpha_{u_0} = (u_0 g)^\natural g^*$.

Next there is proved in [2]:

Let $f_0, f_1\colon L \to X$ be extensions of $u_0\colon K \to X$, then we have: $f_0 \sim f_1$ if and only if $d(f_0, f_1) \in \alpha_{u_0}\pi_1(X^K, u_0)$.

Therefore, if the map $u_0\colon K \to X$ can be extended to $L$, the homotopy classes of extensions of $u_0$ are in $1-1$ correspondence with the elements of $\pi_{r+1}(X, x_0)/\alpha_{u_0}\pi_1(X^K, u_0)$.

In case $L = S_1^n \times S_2^n$, $K = S_1^n \vee S_2^n$, $n > 1$, it turns out that the subgroup

$\alpha_{u_0}\pi_1(X^K, u_0)$ of $\pi_{r+1}(X, x_0)$ consists of the sums of Whitehead products

$$[\omega, \eta]+[\nu, \zeta]$$

where $\eta, \zeta \in \pi_{n+1}(X)$ are arbitrarily chosen, and where $\omega, \nu \in \pi_n(X)$ are represented by the maps $u_0|S_1^n$, $u_0|S_2^n$ respectively.

Let us now return to the case $L=S_1^n \times S_2^n$, $K=S_1^n \vee S_2^n$, $X=S^n$. First we make the following remark:

(7.1) Let $\iota_n$, $\eta_n$ be generators of $\pi_n(S^n)$, $\pi_{n+1}(S^n)$ respectively. Then $2[\iota_n, \eta_n]=0$ when $n\geqslant 3$, for then is $\pi_{n+1}(S^n)\equiv Z_2$.

It is now very easy to deduce from (7.1) and from [2] the following lemma:

(7.2) **Lemma**

Let $n\geqslant 3$ and let there be maps $S^n\times S^n \to S^n$ of type $(p, q)$. Then:

a. The set of homotopy classes of maps $S^n\times S^n\to S^n$ of type $(p, q)$ is in $1-1$ correspondence with the elements of $\pi_{2n}(S^n)/W$, where $W$ contains only the elements $0$, $p[\iota_n, \eta_n]$ and $q[\iota_n, \eta_n]$.
b. Let $f, g: S^n\times S^n\to S^n$ be maps of type $(p, q)$ which agree on $\Sigma$. Then $f$ is homotopic to $g$, if and only if $d(f, g)\in W$.

Consider the suspension homomorphism $E: \pi_{2n}(S^n)\to \pi_{2n+1}(S^{n+1})$.

(7.3) If $n\geqslant 3$, then the kernel of $E$ consists precisely of the elements $0$ and $[\iota_n, \eta_n]$.

The assertion in (7.3) is easily proved by taking $i=2n+1$ in the exact sequence (2.2).

Because of (7.1) and (7.3) we have in lemma (7.2):

(7.4) If $p, q$ are both even, then $\pi_{2n}(S^n)/W$ is isomorphic to $\pi_{2n}(S^n)$. If $p, q$ are *not* both even, then $\pi_{2n}(S^n)/W$ is isomorphic to $E\pi_{2n}(S^n)$.

A. *Homotopy classification of maps $S^n\times S^n\to S^n$ of type $(p, q)$, where $n\geqslant 3$ and $p, q$ are both even.*

In this case the subgroup $W$ of $\pi_{2n}(S^n)$ contains only the element $0$.

Take an arbitrary but fixed map $F: S^n\times S^n\to S^n$ of type $(p, q)$. For every map $f: S^n\times S^n\to S^n$ of type $(p, q)$ we define an element $d_F(f)\in\pi_{2n}(S^n)$ as follows. Let $f'$ be a map such that $f'\sim f$ and $f'|\Sigma=F|\Sigma$. We put

$$d_F(f)=d(F, f').$$

It follows from lemma (7.2) that $d_F(f)$ does not depend on the choice of $f'$ within the homotopy class of $f$.

Furthermore, with the help of (3.8) it is easy to verify that the following lemma is valid:

(7.5) **Lemma**

Let $n \geqslant 3$, and let $p, q$ be both even.

a. Let $F: S^n \times S^n \to S^n$ be a fixed map of type $(p, q)$.
Then the homotopy classes of maps $S^n \times S^n \to S^n$ of type $(p, q($ are fully determined by $d_F$.

b. Let $f, g: S^n \times S^n \to S^n$ be maps of type $(p, q)$. By choosing $F = f$, we have:
$f$ is homotopic to $g$, if and only if $d_f(g) = 0$.

B. *Homotopy classification of maps $S^n \times S^n \to S^n$ of type $(p, q)$, where $n \geqslant 3$ and $p, q$ are not both even.*

In this case it follows from (7.3) that the subgroup $W$ of $\pi_{2n}(S^n)$ in lemma (7.2) is just the kernel of the suspension homomorphism $E$.

Therefore we can rewrite lemma (7.2) (b):

Let $f, g: S^n \times S^n \to S^n$ be maps of type $(p, q)$ which agree on $\Sigma$, then $f \sim g$, if and only if $Ed(f, g) = 0$.

We take again an arbitrary but fixed map $F: S^n \times S^n \to S^n$ of type $(p, q)$. For each map $f$ of type $(p, q)$ we define an element $c_F(f) \in E\pi_{2n}(S^n)$ by:

$$c_F(f) = c(f) - c(F).$$

It follows from (4.5) that $c_F(f)$ is independent of $f$ within its homotopy class.

Let $f, h: S^n \times S^n \to S^n$ be such that $c_F(f) = c_F(h)$. Then $c(f) = c(h)$.

Next we choose a map $f'$ such that $f' \sim f$ and $f'|\Sigma = h|\Sigma$.

From lemma (4.4) we obtain:

$$c(f) - c(h) = c(f') - c(h) = E\,d(h, f') = 0.$$

Hence $f' \sim h$, and thus $f \sim h$.

Therefore we can write down:

(7.6) **Lemma**

Let $n \geqslant 3$, and let $p, q$ be not both even.

a. Let $F: S^n \times S^n \to S^n$ be a fixed map of type $(p, q)$.
Then the homotopy classes of maps $S^n \times S^n \to S^n$ of type $(p, q)$ are fully determined by $c_F$.

b. Let $f, g: S^n \times S^n \to S^n$ be maps of type $(p, q)$.
Then $f$ is homotopic to $g$, if and only if $c(f) = c(g)$.

C. *Homotopy classification of maps $S^n \times S^n \to S^n$ of type $(p, q)$, where $n = 2$ or $n = 4k - 1$ and $p, q$ are arbitrary.*

Let $F: S^n \times S^n \to S^n$ be a fixed map of type $(p, q)$. Let $f, f'$ be maps of type $(p, q)$ such that $f \sim f'$ and $f|\Sigma = f'|\Sigma = F|\Sigma$.

Then, by (4.4) and (4.5), we have:

$$Ed(F, f) = c(f) - c(F) = c(f') - c(F) = Ed(F, f').$$

So $d(F, f) = d(F, f')$, since $E$ is injective by (2.3).

Therefore we can form $d_F(f)$ and $c_F(f)$ for any map $f: S^n \times S^n \to S^n$ of type $(p, q)$, and the relation between them is: $E\, d_F(f) = c_F(f)$.

Let $f, g$ be maps of type type $(p, q)$ such that $f \sim g$, then $c_F(f) = c_F(g)$ by (4.5), so $d_F(f) = d_F(g)$.

For maps $f, g$ let $c_F(f) = c_F(g)$, then $d_F(f) = d_F(g)$, so $f \sim g$ by (3.7).

(7.7) **Lemma**

Let $n = 2$ or let $n = 4k - 1$.

a. Let $F: S^n \times S^n \to S^n$ be a fixed map of type $(p, q)$.
Then the homotopy classes of maps $S^n \times S^n \to S^n$ of type $(p, q)$ are fully determined by $c_F$, as well as by $d_F$.

b. Let $f, g: S^n \times S^n \to S^n$ be maps of the same type type.
Then the following relations are equivalent:

$$f \sim g, \;\; d_f(g) = 0, \;\; c(f) = c(g).$$

To complete this classification, we remark:

(7.8) Let $f, g: S^1 \times S^1 \to S^1$ be maps of the same type.
Then $f$ is homotopic to $g$.

For choose a map $f': S^1 \times S^1 \to S^1$ such that $f' \sim f$ and $f' | \Sigma = g | \Sigma$. Then $d(f', g) = 0$, since $\pi_2(S^1) = 0$. Hence $f' \sim g$ and thus $f \sim g$.

*(To be continued)*